1. Answer the following questions:

   (a) Define the derivative $f'$ of a distribution $f(x)$. Calculate $f'$ and $f''$ if $f = |x|$.

   (b) What is the defining property of the delta distribution?

   (c) If $\hat{L}$ is a self-adjoint operator on $[a,b]$, then $\hat{L}y = f$ with homogeneous boundary conditions, $\hat{B}y = 0$, has a solution if and only if $\int_a^b f y_0 dx = 0$ where $y_0$ is the solution of $\hat{L}y = 0$, $\hat{B}y = 0$. What is the analog of this statement in linear algebra?

   (d) What is the Gibbs phenomenon?

2. Solve:

$$u_t - u_{xx} = 0, \quad 0 < x < \pi, \quad t > 0,$$
$$u(x,0) = 0, \ \ u(0,t) = g(t), \quad u(\pi,t) = 0, \ \ t \geq 0.$$

3. Solve the initial value problem

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x) = \begin{bmatrix} 0 \\ f(x) \end{bmatrix},$$

where $-\infty < x < \infty, \quad t > 0$. Next take $f(x) = e^{-x^2}$ and sketch $u_1(x,t)$ at $t = 0,\ 1,\ 2$. Describe the character of the solution (i.e. of $u_1$ and $u_2$) as $t \to \infty$.

4. Using the Green's identities:

   (a) Prove the uniqueness of the solution (assuming it exists) of the following problem:

   $$\Delta u = f(\mathbf{x}), \quad \mathbf{x} \in D, \quad \mathbf{n} \cdot \nabla u + \alpha(\mathbf{x})\, u = \beta(\mathbf{x}), \quad \mathbf{x} \in B,$$

   where $B$ is the boundary of a bounded domain $D \in \mathbb{R}^3$, $\mathbf{n}$ is the unit normal to $B$, and $\alpha > 0$.

   (b) Is the solution unique when $\alpha = 0$? If not, are there any constraints on the data (i.e. on $f$ and $\beta$) required for the existence of a solution?

5. Find the Fourier transform of $u(x,t)$ if $u$ solves the initial value problem: $u_t = u_{xx} + u_x$, $-\infty < x < \infty$, $t > 0$, $u(x,0) = f(x)$;

   (a) Determine $u$ explicitly when $f(x) = \delta(x)$; explain what this solution means physically.

   (b) Explain the effect of the $u_x$ term.

6. Solve $u_t - uu_x = t$, $t > 0$, $x \in \mathbb{R}$, with $u(x,0) = f(x)$ smooth. Is the solution smooth for all time? If not explain what happens.