KAUST
AMCS241/CS241/EE241 Probability and Random Processes (Fall 2013)

Final Exam (180 minutes)

Date: Tuesday, December 10, 2013.

Code:

Exam: The exam is closed books and closed notes. However you are allowed three sheets of notes (A4 format; both sides). No photocopies allowed.

Partial Credit Policy: Partial credit is awarded **provided you show your work and provided I can decipher it**. In particular it is very important to show clearly all the steps in solving a question. Also answers to question **without work shown or without any explanation will earn no credit**.

Problem 1 (8 points):

We define the function $f_X(.)$ as follows:

$$f_X(x) = \begin{cases} Ke^{ax} & \text{if } x \leq 0, \\ Ke^{-bx} & \text{otherwise,} \end{cases}$$

where $a > 0$, $b > 0$, and K is a positive constant.

a- Find K such that $f_X(.)$ is a valid probability density Function (PDF). **(3 points)**

b- Find the corresponding Cumulative Distribution Function (CDF). **(5 points)**

Problem 2 (10 points):

Let X be a random variable with positive entries. Consider the random variable $Y = 2a[\frac{X}{2}] - aX + 1$, where $[x]$ is the floor function of x and $a \in \mathbb{R}^*$. Find the probability mass function (PMF) of Y and find its mean and variance if X follows a geometric distribution with parameter θ:

$$P(X = k) = (1 - \theta)\theta^k, \ \forall k \in \mathbb{N}.$$

Problem 3 (12 points):

Let (X, Y) be a couple of real random variables having the following joint PDF:

$$f_{X,Y}(x,y) = y - x, \forall x \in]0,1[\text{ and } y \in]1,2[.$$

a- Find the marginal densities of X and Y. **(5 points)**
b- Are X and Y independent? **(2 points)**
c- Find the PDF of $U = X + Y$. **(5 points)**

Problem 4 (25 points) :

Let X, Y and Z be three random variables. Assume that the PDF of X is defined as:

$$f(x) = \frac{1}{16x^4} e^{-\frac{1}{2x}}, \forall x > 0,$$

and (Y given $X = x$) and (Z given $X = x$) are independent and follow the same Gaussian distribution with zero mean and variance equal to x.

a- Find the joint PDF of (Y, Z). **(5 points)**

b- Find the conditional PDF of (X given $Y = y$ and $Z = z$), $f_{X|Y,Z}$. **(5 points)**

c- Find $E[X|Y, Z]$. **(5 points)**

d- Let $U = Y + Z$. Find the PDF of X given $U = u$. **(5 points)**

e- Find $E[X|U]$. **(5 points)**

Problem 5 (10 points) :

A random process is defined by

$$X(t) = T + (1 - t),$$

where T is a uniform random variable over $[0, 1]$. Find

a- The CDF of $X(t)$. **(5 points)**

b- The mean $E[X(t)]$ and the autocovariance, $C_{XX}(t_1, t_2)$, of $X(t)$. **(5 points)**

Problem 6 (7 points) :

Consider the random process $X(t)$ defined as

$$X(t) = Y\cos(\theta t) + Z\sin(\theta t),$$

where Y and Z are two independent centered real random variables having the same variance σ^2 and θ is a real number.

a- Calculate the mean, the variance, and the autocorrelation function of $X(t)$. **(5 points)**

b- Show that $X(t)$ is wide sense stationary. **(2 points)**

Problem 7 (12 points) :
Consider the random process $X(t)$ defined as

$$X(t) = \cos(2\pi t Y + Z),$$

where Y and Z are two independent random variables. The PDF of Y is defined by $f_Y(y)$ while Z is uniform over $]-\pi, \pi[$.
a- Show that X is wide sense stationary. **(5 points)**
b- Find its power. **(2 points)**
c- Find its power spectral density. **(5 points)**

Problem 8 (16 points) :
A random process $X(t)$ is input into a system with the following transfer function:

$$H(j\omega) = 1 + j\omega$$

a- Find the power spectral density $S_Y(w)$ and the autocorrelation function $R_Y(\tau)$ of the system output $Y(t)$ in terms of $S_X(\omega)$ and $R_X(\tau)$. **(8 points)**
b- What is the power of the output. **(3 points)**
c- Find $S_Y(w)$, $R_Y(\tau)$ and the power of $Y(t)$ if $R_X(\tau) = \sigma^2 e^{-|\tau|}$. **(5 points)**