A.M.C.S. 251
Final Examination

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In the following exercises, we denote by $\mathbb{R}^n$ the Euclidean space of dimension n. The canonical (standard) scalar product on $\mathbb{R}^n$ is defined as follows : If $x = (x_1, .., x_n)$ and $y = (y_1, ..., y_n)$, then

$$(x|y) = \sum_{i=1}^{n} x_i y_i.$$

Moreover, if A is any matrix, we denote by ${}^tA$ the transpose of A.

Exercise 1 - 2 points - *Let $\mathbb{R}^3$ be the Euclidean space and let $(.|.)$ be its standard scalar product. Consider the vectors $U = (1, 1, 2), V = (1, 0, 2), W = (2, 1, 4)$.*

1. *Using the Gram-Schmidt algorithm, build a family of orthonormal vectors in the subspace spanned by U, V, W.*
2. *Are these orthonormal vectors an orthonormal basis of $\mathbb{R}^3$?*
3. *How can one deduce from the previous procedure an orthonormal basis of $\mathbb{R}^3$?*

Exercise 2 - 2 points -

1. *Let (e_1, e_2, e_3) be the canonical orthonormal basis of the Euclidean space E^3, and let (ϵ_1, ϵ_2) be the canonical orthonormal basis of $\mathbb{R}^2$. Let*

   $$f : \mathbb{R}^3 \to \mathbb{R}^2$$

   be the linear map defined by

   $$f(e_1) = 2\epsilon_2, f(e_2) = 0, f(e_3) = 0.$$

   What is the kernel (null space) and the rank of f ?
2. *Find the SVD decomposition of the following matrix :*

   $$Z = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Exercise 3 - 3 points - *Consider the following linear system*

$$Ax = b. \tag{1}$$

1. *Suppose* $A = \begin{bmatrix} 5 & 2 \\ 2 & 0 \end{bmatrix}$,
   (a) *Could you use a QR decomposition of A to solve 1 ? Justify your answer.*
   (b) *Could you use the Gauss-Seidel method ? Justify your answer.*
2. *Suppose* $A = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}$.
   (a) *Could you use a LU decomposition of A to solve 1 ? Justify your answer.*
   (b) *Could you use a Cholesky factorization of A to solve 1 ? Justify your answer.*
3. *Suppose* $A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$. *Could you use Jacobi method to solve 1 ? Justify your answer.*

Exercise 4 - 2 points - *Read the following code and answer the questions below.*

```
function [Q,R]=Householder(A)
    [m,n]=size(A);
    if m≠n error('Expect a square matrix'), return, end
    R = A; Q = eye(size(A));
    for k=1:m-1
        i=k-1;j=n-i;v=R(k:n,k);w=v+norm(v)*[1;zeros(j-1,1)];
        Hw=House(w);
        Hk=[eye(i,i) zeros(i,j); zeros(j,i) Hw];
        Q=Hk*Q;
        R(k:n,k:n)=Hw*R(k:n,k:n);
    end
    Q=Q';
end

function H=House(v)
    [n,m]=size(v);
    if m≠1
        error('enter a vector')
    else
        H=eye(n,n);
        nr=norm(v);
    end
    if nr>1.e-10
        H=H -2*v*v'/nr/nr;
    end
end
```

1. *If the input of function* **Housholder** *is* $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$*, what is the output* Q *and* R *?*
2. *What kind of matrix in general are* Q *and* R *?*

Exercise 5 - 4 points - *Let* $\mathbb{R}^n$ *be the Euclidean space of dimension* n*.*

1. *Let* $||.||$ *be any norm on* $\mathbb{R}^n$*. We denote also by* $||.||$ *the subordinate (induced) norm on the space of* (n,n)*-matrices* $\mathcal{M}_n(\mathbb{R})$*. If* U *is an invertible matrix, we denote by* $cond(U)$ *the condition number of* U *with respect to the norm* $||.||$*. Let* A *and* B *be invertible matrices. Compare* $cond(AB)$ *and* $cond(A)cond(B)$*.*
2. *Let* $||.||_{l^2}$ *be the standard Euclidean norm on* $\mathbb{R}^n$*, associated to the standard scalar product. Let* $||.||_2$ *denote the subordinate (induced) norm on the space of* (n,n)*-matrices* $\mathcal{M}_n(\mathbb{R})$*. If* U *is an invertible matrix, we denote by* $cond_2(U)$ *the condition number of* U *with respect to the norm* $||.||_2$*.*
   (a) *Prove that if* A *is an invertible matrix,*
   $$cond_2(A) = \sqrt{\frac{\sigma_{max}}{\sigma_{min}}},$$
   where σ_{max} *denotes the greatest eigenvalue of* ${}^tAA$ *and* σ_{min} *denotes the smallest eigenvalue of* ${}^tAA$*.*
   (b) *Suppose that* $A = \alpha Q$*, where* α *is a non zero real number and* Q *is an orthogonal matrix. Calculate* $cond_2(A)$*.*
   (c) *Suppose that* $cond_2(A) = 1$*. Prove that* $A = \alpha Q$*, where* α *is a non zero real number and* Q *is an orthogonal matrix.*

Exercise 6 - 4 points - *Let* $\mathbb{R}^n$ *be the Euclidean space of dimension* n *and let* $(.|.)$ *be its standard scalar product. Let* $A \in \mathcal{M}_n(\mathbb{R})$ *be a symmetric positive definite matrix. Consider the linear system*

$$Ax = b, \tag{2}$$

where b *is a vector of* $\mathbb{R}^n$*. Let us write the classical decomposition of* A *:*

$$A = D - E - F,$$

where D *is the diagonal of* A*,* $-E$ *is the lower triangular matrix whose terms are under the diagonal of* A *and* $-F$ *is a upper triangular matrix whose terms are above the diagonal of* A*. Let us define two sequences* $(x^{(k)})$ *and* $(\tilde{x}^{(k)})$ *of points of* $\mathbb{R}^n$ *:* $x_0 = x^{(0)}$*,* $\tilde{x}_0 = \tilde{x}^{(0)}$*,*

$$\begin{cases} x^{(k+1)} & = \frac{1}{3}\tilde{x}^{(k+1)} + \frac{2}{3}x^{(k)}, \\ D\tilde{x}^{(k+1)} - Ex^{(k+1)} & = Fx^{(k)} + b. \end{cases}$$

1. *Find matrices M and N such that for all k,*

$$Mx^{(k+1)} = Nx^{(k)} + b.$$

2. *Prove that the sequence $(x^{(k)})$ converges to the solution of the linear system 2.*

   One can use (without proof) the following well known result, given in the course : Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric positive definite matrix. If A admits a decomposition $A = M - N$ with M invertible and ${}^tM + N$ symmetric positive definite, then $\rho(M^{-1}N) < 1$, where $\rho(M^{-1}N)$ denotes the spectral radius of $M^{-1}N$.

Exercise 7 - 3 points - *Let $\mathbb{R}^n$ be the Euclidean space of dimension n and let $(.|.)$ be its standard scalar product. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable matrix.*

1. *Describe the power method which approximates the eigenvalue λ_1 of A with maximal module and a corresponding eigenvector U_1. Give a sufficient condition on the initial condition which implies that the algorithm converges.*
2. *Suppose that A is symmetric positive definite. Find an endomorphism*

$$g : \mathbb{R}^n \to \mathbb{R}^n,$$

   of rank $n-1$ satisfying the following property : for every X orthogonal to U_1, $g(X)$ is orthogonal to U_1.
3. *Deduce a method which allows to compute the second eigenvalue λ_2 of A, that is, the largest eigenvalue after λ_1.*

Exercise 8 - 2 points - *Let $\mathbb{R}^n$ be the Euclidean space of dimension n and let $(.|.)$ be its standard scalar product. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric positive definite matrix. Consider the function*

$$f : \mathbb{R}^n \to \mathbb{R},$$

defined by

$$f(x) = \frac{1}{2}{}^txAx - {}^tbx, \tag{3}$$

where b is a vector of $\mathbb{R}^n$. We apply the Conjugate Gradient method to minimize f. Let $x^{(0)}$ be an initial condition and let $x^{(k)}$ be the sequence of points associated to the conjugate gradient algorithm by the standard construction :

$$x^{(k+1)} = x^{(k)} + \rho_k w^{(k)},$$

where we suppose that the $w^{(k)}$ are well defined and different to 0 for $0 \leq k \leq n$. Evaluate the matrix

$$\left(\frac{(Aw^{(i)}|w^{(j)})}{(Aw^{(i)}|w^{(i)})}\right)_{1\leq i\leq n-1, 1\leq j\leq n-1}.$$