MCSE Core Qualifying Exam: PDEs **19 May 2012**

For full credit, answer all questions in both parts #1 – #2 (both sides of the page). Attach extra work as needed and copy your final answers into the blanks on this page.

1) Consider the differential equation for $u(x, y)$: $\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 1$.

(a) Classify this equation as to linearity (or not), order, and type (elliptic, hyperbolic, parabolic).

(b) Solve the equation subject to $u(0, y) = y$.

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1) Consider a family of Runge-Kutta integrators for an ODE $y' = f(x, y)$.

(a) For a given stepsize h in the coordinate x, these integrators commit a local truncation error $\propto h^p$. For a first-order Euler method ($p = 1$) there is one evaluation of $f(x, y)$ per step. For a fourth-order Runge-Kutta method ($p = 4$) there are four evaluations of $f(x, y)$ per step, but the steps can be larger for the same level of local truncation error. A first-order method is used with $h = \frac{1}{256}$, taking 256 steps over the interval (0,1) and evaluating the function f 256 times. For roughly the same local truncation error, how many steps does the fourth-order method need to take, and how times altogether does the fourth-order method evalute f?

(b) Consider the following means for using two or more computational approximations of the solution of an ODE, without changing the order of the method, to improve the accuracy of the desired output beyond what one could achieve from any individual computational approximation:

Let the true result of integrating the ODE from an initial t_0 to a final time t_f be called u. Let the computational approximation to u when integrated with timestep h be called $\tilde{u}_h$. There will be computational error, called e_h, such that $u = \tilde{u}_h + e_h$. Though we do not know the error, we know the form of the dominant term of the error: $e_h \approx c \cdot h^p$, where p is the order of the method, but c is an unknown constant. Information from extra computations at different stepsizes can be used to estimate c.

Suppose we integrate the same ODE from t_0 to t_f using two different stepsizes: h and h/r, obtaining results $\tilde{u}_h$ and $\tilde{u}_{h/r}$, respectively. Then we may write:

$$u \approx \tilde{u}_h + c \cdot h^p$$

$$u \approx \tilde{u}_{h/r} + c \cdot (\frac{h}{r})^p$$

Derive an estimate for u that involves both computational outputs, $\tilde{u}_h$ and $\tilde{u}_{h/2}$, and the known exponent p and refinement ratio r, from which c is eliminated.