Applied Mathematics Qualifying Exam: PDEs **21 May 2011**

1) Wave equation in a finite domain

Consider the PDE initial-boundary value problem for $u(x,t)$:

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f(x,t)\\ u(0,t) &= 0\\ u(1,t) &= 0\\ u(x,0) &= g(x)\end{aligned}$$

The left-hand side operator has spatial eigenfunctions $\phi_n(x)$ with eigenvalues λ_n satisfying the boundary conditions, which allows a solution to be written by superposition:

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\phi_n(x),$$

where the $A_n(t)$ are the time-varying amplitudes of each spatial mode. If $f(x,t) = 0$ for $t > 0$, then the amplitudes are set by the initial condition $g(x)$ and are fixed throughout the evolution, allowing a solution to be written

$$u(x,t) = \sum_{n=1}^{\infty} a_n \psi_n(t)\phi_n(x),$$

where the $\psi_n(t)$ carry the time dependence, as the separation companions to the spatial eigenfunctions.

(a) Find $\phi_n(x)$ and $\psi_n(t)$ for the the given problem.

(b) For $f(x,t) = 0$ and general $g(x)$, give a formula for the a_n, $n = 1, 2, \ldots$, exploiting the orthogonality of the spatial eigenfunctions.

(c) For $g(x) = 0$ and general $f(x,t)$, give a set of ODEs in time for the $A_n(t)$, $n = 1, 2, \ldots$, exploiting the orthogonality of the spatial eigenfunctions. Specify the initial conditions for the ODEs.

(d) Finally, let $f(x,t) = \sin(\pi x)$ and $g(x) = \sin(17\pi x)$ and write down the solution to the entire problem.

2) Canonical form for second-order PDE

Consider the differential equation for $u(x,y)$:

$$y^2 u_{xx} - x^2 u_{yy} = 0.$$

(a) Classify this equation.

(b) Find the ordinary differential equations for the characteristic curves.

(c) Find new variables in which the PDE has canonical form.

(d) Transform the PDE to canonical form. (You are *not* required to integrate it.)