1) Subspace projections

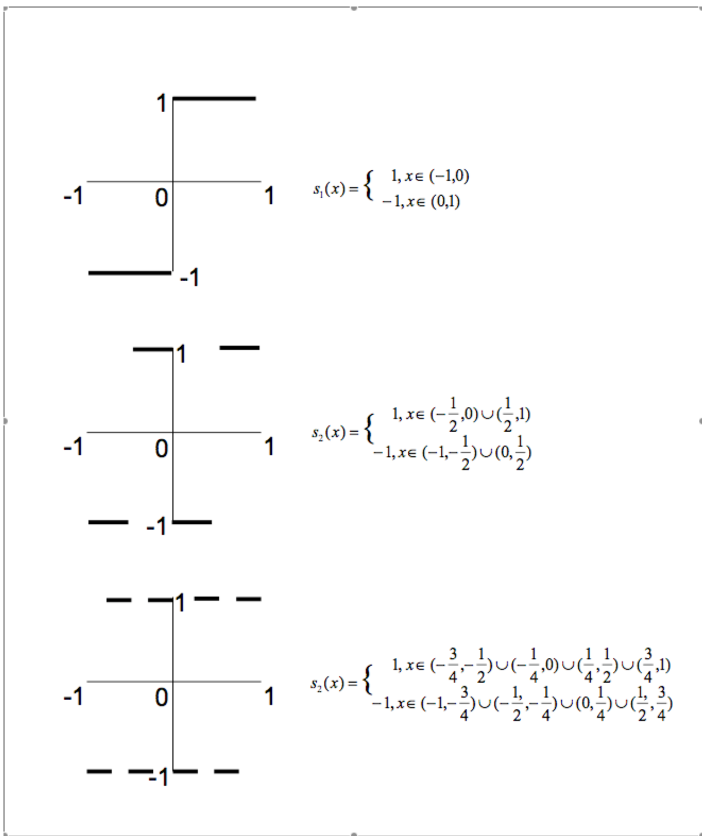
Consider the subspace S_N of L_2 of piecewise constant ("staircase") functions spanned by the first N "square-wave sinusoids," $\{s_1(x), s_2(x), \ldots, s_N(x)\}$, defined on the interval $[-1, 1]$ as in the figure.

(a) Project the function $f(x) = x$ onto S_N for $N = 2$ in the L_2 sense. That is, find the coefficients of the expansion $P_N f(x) \equiv f_N(x) = a_1 s_1(x) + a_2 s_2(x) + \ldots + a_N s_N(x)$, such that $(f - P_N f, \chi) = 0$ for $\chi \in S_N$. Sketch the resulting function, $f_N(x)$.

(b) Repeat for $f(x) = x$ and $N = 3$.

(c) Repeat for $f(x) = x^2$ and $N = 2$.

(d) Comment (very briefly) on the sufficiency (or lack thereof) of this basis for representing general infinitely smooth functions on $[-1, 1]$. How would you recommend improving it, in at least two conceptually different ways?



2) Stability of the beam equation

Consider the transient deflection $u(x, t)$ of an elastic beam with constant material properties:

$$u_t = -u_{xxxx},$$

semi-discretized in space on the real line with uniform mesh spacing h, with second-order central differences. Using the approximation

$$u_{xxxx}(x, t) = \frac{1}{h^4}[u(x - 2h, t) - 4u(x - h, t) + 6u(x, t) - 4u(x + h, t) + u(x + 2h, t)] + \mathcal{O}(h^2),$$

find the explicit stability limit on the timestep, k, for a first-order forward Euler scheme in time.