KAUST
AMCS241/CS241/EE241 Probability and Random Processes (Fall'11)
Final Exam (170 minutes)

Date: Saturday 10^{th} December 2011.

Name:

Exam: The exam is closed books and closed notes. However you are allowed three sheet of notes (A4 format; both sides). No photocopies allowed.

Partial Credit Policy: Partial credit is awarded **provided you show your work and provided I can decipher it**. In particular it is very important to show clearly all the steps in solving a question. Also answers to question **without work shown or without any explanation will earn no credit**.

Problem 1(5 points):

Suppose the variance of an IID sequence of random variables is formed according to

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{m=1}^{n}(X_m - \hat{\mu})^2,$$

where $\hat{\mu}$ is the sample mean. Find the expected value of this estimate and show that it is biased.

Problem 2(5 points):

The relation between the input $X(t)$ and output $Y(t)$ of a diode is expressed as

$$Y(t) = X^2(t).$$

Let $X(t)$ be a zero-mean stationary Gaussian random process with autocorrelation

$$R_X(\tau) = e^{-\alpha|\tau|}, \qquad \alpha > 0.$$

(a) Find the output mean $\mu_Y(t) = E[Y(t)]$.
(b) Find the autocorrelation $R_Y(t, t+\tau)$.
(c) Is $Y(t)$ wide sense stationary?

Problem 3(7 points):

Let $X_1,\ X_2,\ldots,\ X_n$ be independent zero mean, unit-variance Gaussian random variables. Let $Y_k = (X_k + X_{k-1})/2$, that is, Y_k is the moving average of the pairs of values of X. Assume $X_{-1} = 0 = X_{n+1}$. Find the covariance matrix of Y_k's.

Problem 4(7 points):

Let X be a random variable with pdf

$$f_X(x) = \begin{cases} 0, & x < 0 \\ ce^{-2x}, & x > 0 \quad c > 0. \end{cases}$$

(a) Find c.
(b) Let $a > 0,\ x > 0$, find $P[X > x + a]$.
(c) Let $a > 0,\ x > 0$, find $P[X > x + a | X > a]$.

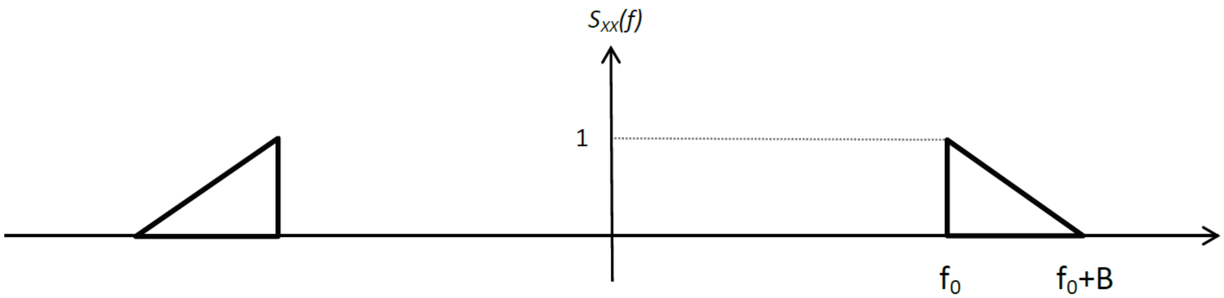
Problem 5(12 points):
A random process is defined by $X(t) = \exp(-At)u(t)$, where A s a random variable with pdf, $f_A(a)$.
(a) Find the pdf of $X(t)$ in terms of $f_A(a)$.
(b) If A is an exponential random variable, with $f_A(a) = e^{-a}u(a)$, find $\mu_X(t)$ and $R_{X,X}(t_1, t_2)$. Is the process WSS?

Problem 6(18 points):

Let $X(t)$ be a random process whose PSD is shown in the accompanying figure. A new process is formed by multiplying $X(t)$ by a carrier to produce

$$Y(t) = X(t)\cos(\omega_0 t + \theta).$$

where θ is uniform over $[0,\ 2\pi]$ and independent of $X(t)$. Find and sketch the PSD of the process $Y(t)$.



Problem 7(14 points):

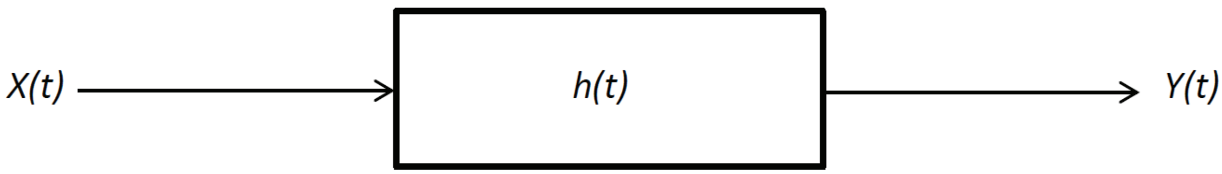
A white Gaussian noise process $N(t)$ with autocorrelation $R_N(\tau) = \alpha\delta(\tau)$ is passed through an integrator yielding the output

$$Y(t) = \int_0^t N(u)du$$

Find $E[Y(t)]$ and the autocorrelation function $R_Y(t, \tau)$. Show that $Y(t)$ is a non-stationary process.

Problem 8(14 points):

Consider the system shown in figure, with impulse response $h(t) = e^{-t}u(t)$. Determine the mean and variance of $Y(t)$ if the input $X(t)$ is a zero mean white noise process.



Problem 9(18 points):

A wide sense stationary (WSS) Gaussian random process $x(t)$ has the following characteristics:

$$E[x(t)] = 0 \ \ \forall \ t$$

$$R_x(t_1, t_2) = \cos(t_1 - t_2) \ \ \forall \ t_1, \ t_2$$

(a) Explain why $R_x(t, t+\tau)$ is a valid autocorrelation function.
(b) Find the power spectral density $S_x(f)$.
(c) Find the power P_x of $x(t)$.
(d) Let $n(t)$ be a deterministic signal and consider the random variable X defined by

$$X = \int_0^T x(t)n(t)dt,$$

where T is a constant. Show that the variance of X is given by

$$\sigma^2 = \int_0^T \int_0^T n(s)n(t)\cos(s-t)dsdt$$

(e) Consider now the random variable Y defined by

$$X = c\int_0^T x(t)dt,$$

where T is a constant and c is a random variable independent of $x(t)$. Find the variance of Y, if c is uniformly distributed over [0,1].